Feasibility of the Mixed Postman Problem with Restrictions on the Edges

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The Mixed Postman Problem

Let $M = (V, E, A)$ be a strongly connected mixed graph and let $c \in \mathbb{Z}^{E \cup A}_+$. 

The mixed postman problem is to find the minimum cost of a tour of $M$ traversing all of its edges and arcs.

This problem is NP-hard, even if restricted to planar inputs.

Feasibility can be decided in polynomial time.
Restrictions on the Edges

We are interested on tours that use edges once.

Figure 1: An arcs postman tour.
Complexity of Feasibility

- Feasibility is NP-complete.
- It remains NP-complete for planar inputs.
- Feasibility can be decided in poly-time if
  - $M$ has total degree even at every vertex,
  - $M$ is series-parallel, or
  - $A$ is a forest.
- Optimization in poly-time too.
- We are interested in necessary conditions (which can be verified in poly-time).
Even Set Condition

- For $S \subseteq V$, let $\delta_M(S')$ be the set of edges and arcs between $S$ and $\tilde{S} = V \setminus S$.
- We say that $S$ is undirected if $\delta_M(S') \subseteq E$.
- If $M$ has an arcs postman tour and $S'$ is undirected then $|\delta_M(S')|$ must be even.
- This condition (due to Veerasamy) can be verified in polynomial time.
Outgoing Set Condition

- We say that $S$ is **outgoing** if all arcs in $\delta_M(S)$ go from $S$ to $\bar{S}$.
- If $M$ has an arcs postman tour and $S$ is outgoing then
  \[ |\delta_M(S) \cap E| \geq |\delta_M(S) \cap A|. \]
- This condition (also due to Veerasamy) can be verified in polynomial time.
Independency and Sufficiency

- $K_2$ satisfies the *outgoing set* condition but *not* the even set condition.
- $\vec{K}_2$ satisfies the *even set* condition but *not* the outgoing set condition.
- These conditions are not sufficient:
Two Outgoing Sets Condition

- For outgoing $S$ we define the \textit{surplus} of $S$ as

$$\text{sur}_M(S) = |\delta_M(S) \cap E| - |\delta_M(S) \cap A|.$$

- \textbf{Theorem:} Let $M$ have an arcs postman tour and let $S, T$ be outgoing. Then

$$|E(S \setminus T, T \setminus S)| \leq \left\lfloor \frac{1}{2}\text{sur}_M(S) \right\rfloor + \left\lfloor \frac{1}{2}\text{sur}_M(T) \right\rfloor.$$

- This condition can be verified in polynomial time (nicer algorithm for planar $M$).
Strength and Sufficiency

- Our condition implies and is stronger than
  - the even set condition (take $T = \bar{S}$) and
  - the outgoing set condition (take $T = S$).
- Our condition is still not sufficient:
Many Outgoing Sets

• Let $S = \{S_1, \ldots, S_k\}$ be $k$ outgoing subsets:

$$|E^S_k| \leq \sum_{i=1}^{k} \lfloor \frac{1}{2} \text{sur}_M(S_i) \rfloor$$

where $E^S_k$ is a certain subset of edges.

• Conjecture: There exists $f : \mathbb{N} \rightarrow \mathbb{N}$ for which:
A connected mixed graph $M$ has an arcs postman tour iff it satisfies the $k$-outgoing-sets condition for all $1 \leq k \leq f(|V(M)|)$.
The mixed graph $M_n$ satisfies the $k$-outgoing-sets condition for each $1 \leq k \leq 2n - 1$, but does not satisfy the $2n$-outgoing-sets condition.

Figure 2: The mixed graph $M_5$. 

\[ \begin{align*}
  &v_0 & e_0 & v_2 & e_2 & v_4 & e_4 & v_6 & e_6 & v_8 & e_8 & v_{10} \\
  &v_1 & e_1 & v_3 & e_3 & v_5 & e_5 & v_7 & e_7 & v_9 & & &
\end{align*} \]
Characterization

The following statements are equivalent:

- $M$ has an arcs postman tour for every orientation of its edges.
- The only outgoing sets of $M$ are $\emptyset$ and $V$.
- $D = (V, A)$ is strongly connected.
- For all $k \in \mathbb{N}$, and for all families $S$ of $k$ outgoing sets, $M$ satisfies:

$$\sum_{e \in E} m^+_S(e) \leq \sum_{i=1}^{k} \left\lfloor \frac{1}{2} \text{sur}_M(S_i) \right\rfloor.$$
Conclusions

• New necessary conditions for feasibility.
• Some of them can be verified in poly-time.
• Can we verify some others too?
• Are our conditions sufficient?
• We are searching for other classes of mixed graphs for which we can decide feasibility or optimize in polynomial time.